ON THE EXISTENCE OF POSITIVE SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. We present some results on the existence of bounded positive solutions to a class of nonlinear second order ordinary differential equations by using the Schauder-Tikhonov fixed point theorem. An application to the existence of bounded positive solutions to certain quasilinear elliptic equations in two-dimensional exterior domains is also given.

1. Introduction

We consider the second order nonlinear ordinary differential equation

\[ u'' + \frac{u'}{t} + f(t, u, u') = 0, \quad t \geq 1, \]

where the nonlinear function \( f : [1, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous.

The goal of the paper is to prove the existence of positive bounded solutions to (1.1) under general conditions on the nonlinear function \( f \). Problems of this type were already considered several decades ago (see e.g. [1], [10] and the citations therein) but this is still a very active area of research, cf. the discussions in [4], [9], [12]. Our results are obtained by applying the Schauder-Tikhonov theorem to an integral form of (1.1). While the fixed point approach to this type of problems is extensively used in the mathematical literature, by working in function spaces where the derivative is also involved, we are able to obtain a considerable improvement with respect to previous works. In particular, our results cover more general situations than the recent studies [4], [8], [9], [12], [17]. Moreover, our main result can be applied to show the existence of bounded positive solutions to certain quasilinear elliptic equations in two-dimensional exterior domains, improving and enhancing some earlier investigations (see [3], [11], [13]).

2. Main results

In this section, we shall prove the existence of bounded positive global solutions to (1.1) under certain conditions on the nonlinearity \( f \).

We first introduce a function space and present two propositions which are used in the proof of Theorem (2.3) in the sequel. Define the set

\[ X = \{ u \in C^1([1, \infty), \mathbb{R}) : \lim_{t \to \infty} u(t) \text{ exists and } \lim_{t \to \infty} u'(t) \text{ exists} \}. \]

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endowed with the usual linear operation for $C^1$ function. Then, we have the following two propositions:

**Proposition (2.1).** The space $X$ is Banach space under the norm
\[ \| u \| = \max \left\{ \sup_{t \geq 1} |u(t)|, \sup_{t \geq 1} |u'(t)| \right\}. \]

**Proof.** Obviously, $X$ is a linear space, so we only need to prove its completeness. By means of the completeness of $C([1, \infty), \mathbb{R})$ endowed with the supremum norm and the integral representation of $u(t)$, the statement follows at once (see also Proposition 10 in [9]).

**Proposition (2.2).** Let $K$ be subset of $X$. Then $K$ is relatively compact in $X$ if and only if the following three properties hold:

(i) $K$ is bounded, that is, there exists a number $M > 0$ such that for all $t \geq 1$ and all $u \in K$,
\[ |u(t)| \leq M \quad \text{and} \quad |u'(t)| \leq M; \]

(ii) $K$ is equicontinuous, that is, for every $\varepsilon > 0$, there exists a $\sigma > 0$ such that for all $|t_1 - t_2| < \sigma$ and all $u \in K$,
\[ |u(t_1) - u(t_2)| \leq \varepsilon \quad \text{and} \quad |u'(t_1) - u'(t_2)| \leq \varepsilon; \]

(iii) $K$ is equiconvergent, that is, for every $\varepsilon > 0$, there exists a $t_\varepsilon > 1$ such that for all $t, s \geq t_\varepsilon$ and all $u \in K$,
\[ |u(t) - u(s)| \leq \varepsilon \quad \text{and} \quad |u'(t) - u'(s)| \leq \varepsilon. \]

**Proof.** The statement is a consequence of the Arzela-Ascoli theorem (see also Proposition 12 in [9]).

We present now a result on the existence of bounded positive solutions to the equation (1.1).

**Theorem (2.3).** Assume that $f$ satisfies the inequality
\[ |f(t, u, u')| \leq F(t, |u|, |u'|), \]
where $F \in C([1, \infty) \times [0, \infty) \times [0, \infty), [0, \infty))$ and $F(t, r, s)$ is nondecreasing in both $r$ and $s$ for each fixed $t \in [1, \infty)$.

(i) If $F$ satisfies for some $c > 0$,
\[ \int_{1}^{\infty} t \max\{1, \ln t\} F(t, 2c, c)dt < c, \]
then there exists a $\delta \in (0, c)$ such that (1.1) has at least a bounded positive solution $u_c$ which satisfies $\delta \leq u_c(t) \leq 2c - \delta$ for $t \geq 1$, and such that $\lim_{t \to \infty} u_c(t)$ exists.

(ii) If $F$ satisfies for some $c > 0$,
\[ \int_{1}^{\infty} t \ln t F(t, 2c, c)dt < \infty, \]
then there exist $t_0 \geq 1$ and $\delta \in (0, c)$ such that (1.1) has at least a bounded positive solution $u_c$ which satisfies $\delta \leq u_c(t) \leq 2c - \delta$ for $t \geq t_0$, and such that $\lim_{t \to \infty} u_c(t)$ exists.
Proof. We first prove part (i). Using the hypothesis (2.5) and the monotonicity property of the function $F$, by applying the Lebesgue dominated convergence theorem we obtain that there exists an $\delta = \delta(c) \in (0, c)$ such that

\begin{equation}
\int_1^\infty t \max\{1, \ln t\} F(t, 2c - \delta, c - \delta) dt \leq c - \delta.
\end{equation}

Set $K = \{v \in X : \delta \leq v(t) \leq 2c - \delta, |v'(t)| \leq c - \delta\}$. Define the operator $T : K \to X$ by

\begin{equation}
(Tv)(t) = c + \ln t \int_t^\infty s f(s, v, v') ds + \int_1^t s \ln s f(s, v, v') ds,
\end{equation}

for $t > 1$, with $(Tv)(1) = c$. We shall apply the Schauder-Tikhonov theorem to prove that there exists a fixed point for the operator $T$ in the nonempty closed bounded convex set $K$.

(1) We check that $T : K \to K$. Note that from the inequalities (2.4) and (2.7) and the monotonicity property of $F$, we have that for every $v \in K$,

\begin{equation}
| (Tv)(t) - c | = | \ln t \int_t^\infty s f(s, v, v') ds + \int_1^t s \ln s f(s, v, v') ds |
\end{equation}

\begin{equation}
\leq \int_t^\infty (\ln t) s \ln s | f(s, v, v') | ds + \int_1^t s \ln s | f(s, v, v') | ds
\leq \int_1^\infty s \ln s F(s, 2c - \delta, c - \delta) ds \leq c - \delta, \quad t \geq 1.
\end{equation}

It follows that $\delta \leq (Tv)(t) \leq 2c - \delta$ for $t \geq 0$. On the other hand, differentiating both sides of (2.8) with respect to $t$, we get

\begin{equation}
(Tv)'(t) = \frac{1}{t} \int_t^\infty s f(s, v, v') ds, \quad t \geq 1.
\end{equation}

Thus, by (2.7) and (2.10),

\begin{equation}
| (Tv)'(t) | = \frac{1}{t} \int_t^\infty s f(s, v, v') ds \leq \int_t^\infty s F(s, v, v') ds
\leq \int_1^\infty s F(s, 2c - \delta, c - \delta) ds \leq c - \delta, \quad t \geq 1.
\end{equation}

It follows that $| (Tv)'(t) | \leq c - \delta$. Thus, $T : K \to K$ is well-defined.

(2) We check that $T(K)$ is relatively compact in $X$. If $\{v_n\}_{n \geq 1}$ is an arbitrary sequence in $K$, set $M = c - \delta$. We have by (2.11) that

\begin{equation}
| (Tv_n)'(t) | \leq c - \delta = M, \quad t \geq 1, \quad n \geq 1.
\end{equation}

An application of the mean value theorem yields

\begin{equation}
| (Tv_n)(t_1) - (Tv_n)(t_2) | \leq M | t_1 - t_2 |, \quad t_1, t_2 \geq 1, \quad n \geq 1.
\end{equation}
On the other hand, from (2.10) we infer that for $t_2 > t_1 \geq 1$,
\[
| (T_{v_n})'(t_1) - (T_{v_n})'(t_2) | = \left| \frac{1}{t_1} \int_{t_1}^{\infty} s f(s, v_n, (v_n)')ds - \frac{1}{t_2} \int_{t_2}^{\infty} s f(s, v_n, (v_n)')ds \right|
\leq \left| \frac{1}{t_1} - \frac{1}{t_2} \right| \int_{t_1}^{\infty} sF(s, v_n, (v_n)')ds + \frac{1}{t_2} \int_{t_1}^{t_2} s f(s, v_n, (v_n)')ds
\leq | t_2 - t_1 | \int_{t_1}^{\infty} sF(s, 2c, c)ds + \int_{t_1}^{t_2} s F(s, 2c, c)ds.
\]
Taking into account (2.13), by (2.5) and the above inequality, we infer that \( \{T_{v_n}\}_{n \geq 1} \) is equicontinuous in \( X \).

Note that
\[
\int_{1}^{t} | s \ln s f(s, v, v') | ds \leq \int_{1}^{\infty} s \ln s F(s, 2c - \delta, c - \delta)ds \leq c - \delta,
\]
so that \( \lim_{t \to \infty} \int_{1}^{t} s \ln s f(s, v, v')ds \) exists. Set
\[
\alpha = \lim_{t \to \infty} \int_{1}^{t} s \ln s f(s, v, v')ds.
\]
By (2.8), we have
\[
(2.14) \quad | (Tv)(t) - c - \alpha | = | \ln t \int_{t}^{\infty} s f(s, v, v')ds + \int_{1}^{t} s \ln s f(s, v, v')ds - \alpha |.
\]
This shows that for every \( \varepsilon > 0 \) there exists \( t_0(\varepsilon) > 1 \) such that
\[
(2.15) \quad | (Tv_n)(t) - c - \alpha | \leq \varepsilon, \quad t \geq t_0(\varepsilon), \quad n \geq 1.
\]
Since by (2.10) we have
\[
| (T_{v_n})'(t) | \leq \int_{t}^{\infty} sF(s, 2c - \delta, c - \delta)ds, \quad t \geq 1,
\]
we deduce that for every \( \varepsilon > 0 \) there exists \( t_1(\varepsilon) > 1 \) such that
\[
(2.16) \quad | (T_{v_n})'(t) | \leq \varepsilon, \quad t \geq t_1(\varepsilon), \quad n \geq 1.
\]
The relations (2.15) and (2.16) show that \( \{T_{v_n}\}_{n \geq 1} \) is equiconvergent in \( X \).

Since \( T_{v_n} \in K \), we also know that \( \{T_{v_n}\}_{n \geq 1} \) is bounded in \( X \). Thus, applying Proposition (2.2), we obtain that \( \{T_{v_n}\}_{n \geq 1} \) is relatively compact in \( X \).

(3) We check that \( T : K \to K \) is continuous. Fix an \( \varepsilon > 0 \). In view of (2.5), there exists some \( t_\ast > 1 \) such that
\[
(2.17) \quad \int_{t_\ast}^{\infty} s \max\{1, \ln s\} F(s, 2c, c)ds < \frac{\varepsilon}{3}.
\]
Since \( f : [1, t_\ast] \times [\delta, 2c - \delta] \times [-(c - \delta), (c - \delta)] \to \mathbb{R} \) is uniformly continuous, there exists an \( \sigma > 0 \) such that
\[
(2.18) \quad | f(\tau, r_1, s_1) - f(\tau, r_2, s_2) | < \frac{\varepsilon}{3(\frac{1}{2}t_\ast^2 \ln t_\ast + \frac{1}{4}t_\ast^2 + \frac{1}{4})}
\]
for all $\tau \in [1, t_{*}]$, all $r_{1}, r_{2} \in [\delta, 2c - \delta]$ with $r_{1} - r_{2} < \sigma$, and all $s_{1}, s_{2} \in\{-(c - \delta), 0, (c - \delta)\}$ with $s_{1} - s_{2} < \sigma$. A straightforward computation using (2.8) shows that, for all $v_{1}, v_{2} \in K$ with $\|v_{1} - v_{2}\| < \sigma$, we have
\[
| (T_{v_{1}})(t) - (T_{v_{2}})(t) | \leq \int_{1}^{\infty} s \max\{1, \ln s\} | f(s, v_{1}, v'_{1}) - f(s, v_{2}, v'_{2}) | ds
\]
\[
\leq \int_{t_{*}}^{\infty} s \max\{1, \ln s\} | f(s, v_{1}, v'_{1}) - f(s, v_{2}, v'_{2}) | ds
\]
\[
+ \int_{t_{*}}^{\infty} s \max\{1, \ln s\} \left[ | f(s, v_{1}, v'_{1}) | + | f(s, v_{2}, v'_{2}) | \right] ds
\]
\[
\leq 3 \left( \frac{4}{3} \ln t_{*} + \frac{1}{2} \ln t_{*} + \frac{1}{4} \right) \int_{1}^{t_{*}} s^{1} (\ln s + 1) s ds + 2 \int_{t_{*}}^{\infty} s \max\{1, \ln s\} F(s, 2c, c) ds
\]
\[
\leq \frac{e}{3} + \frac{2e}{3} = e,
\]
in view of (2.17)-(2.18). Similarly, from (2.10) we infer that
\[
| (T_{v_{1}})'(t) - (T_{v_{2}})'(t) | \leq \frac{1}{t} \int_{t}^{\infty} s | f(s, v_{1}, v'_{1}) - f(s, v_{2}, v'_{2}) | ds
\]
\[
\leq \int_{1}^{\infty} s \max\{1, \ln s\} | f(s, v_{1}, v'_{1}) - f(s, v_{2}, v'_{2}) | ds \leq e.
\]
It follows that
\[
\| T_{v_{1}} - T_{v_{2}} \| \leq \max\{ \sup_{t \geq 1} | T_{v_{1}}(t) - T_{v_{2}}(t) |, \sup_{t \geq 1} | (T_{v_{1}})'(t) - (T_{v_{2}})'(t) | \} \leq e.
\]

Hence, $T : K \to K$ is a continuous operator.

We have verified that $T : K \to K$ satisfies all assumptions of the Schauder-Tikhonov theorem [6]. Hence there exists $u_{c} \in K$ such that $Tu_{c} = u_{c}$. Therefore $\delta \leq u_{c}(t) \leq 2c - \delta$ for $t \geq 1$, and
\[
u_{c}(t) = c + \ln t \int_{t}^{\infty} s f(s, u_{c}(s), u'_{c}(s)) ds + \int_{1}^{t} s \ln s f(s, u_{c}(s), u'_{c}(s)) ds, t \geq 1.
\]
Differentiating both sides of the above equation with respect to $t$, we get that $u_{c}(t)$ is a solution to (1.1) which satisfies
\[
\lim_{t \to \infty} (u_{c}(t) - c - \alpha) = 0,
\]
in view of (2.14). This proves part (i) of Theorem (2.3).

Let us now prove part (ii) of Theorem (2.3). In view of (2.6), if we take $t_{0} \geq 1$ sufficiently large, then we have
\[
\int_{t_{0}}^{\infty} t \max\{1, \ln t\} F(t, 2c, c) dt < c.
\]

For the equation (1.1) on $[t_{0}, \infty)$, set
\[
X_{1} = \{ v \in C^{1}([t_{0}, \infty), \mathbb{R}) : \lim_{t \to \infty} v(t) \text{ exists and } \lim_{t \to \infty} \{v'(t)\} \text{ exists} \},
\]
and let $K_1 = \{ v \in X_1 : \delta \leq v(t) \leq 2c - \delta, \ | v'(t) | \leq c - \delta, \ \text{for} \ t \geq t_0 \}$. Define the operator $T_1 : K_1 \to X_1$ by 
\[
(T_1 v)(t) = c + \frac{\ln t}{t_0} \int_0^{\infty} s f(s, v, v') \, ds + \int_{t_0}^t s \ln s f(s, v, v') \, ds, \quad t > t_0,
\]
with $(T_1 v)(t_0) = c$.

Following the proof for the part (i) of Theorem (2.3), we verify that the operator $T_1 : K_1 \to K_1$ satisfies all assumptions of the Schauder-Tikhonov theorem. Thus there exists a fixed point for the operator $T_1$ in the nonempty closed bounded convex set $K_1$ and the statement (ii) of Theorem (2.3) is true. This completes the proof.

**Example (2.19).** Consider (1.1) with $f(t, u, u') = q(t) g(u)$, where $g \in C(\mathbb{R}, \mathbb{R})$ and $q \in C([1, \infty), [0, \infty))$. Theorem (2.3) guarantees the existence of a solution to (1.1) that is bounded and positive in a neighborhood of infinity if 

\[(2.20) \quad \int_1^\infty t q(t) \ln(t) \, dt < \infty.\]

For $q(t) = t^\beta$, $t \geq 1$, note that (2.20) holds if and only if $\beta < -2$. To see that this result is sharp, it suffices to restrict our attention to the particular case $g(u) \equiv u$. Then the substitution $v(t) = \sqrt{t} u(t)$ transforms (1.1) into 

\[(2.21) \quad v'' + \left( t^\beta + \frac{1}{4t^2} \right) v = 0, \quad t \geq 1.\]

It is known (see [7], page 461) that the necessary and sufficient condition for the existence of a solution to (2.21) that is positive in a neighborhood of infinity is precisely $\beta < -2$. Let us also note that the condition (2.20), the sharpness of which we just established, can not be obtained in the same general setting as above by using the results in [4], [5], [8], [9], [17].

**Remark.** The existence of non-oscillatory solutions to (1.1) has also been recently investigated in [9]. However, the approach devised in [9] relies on the use of nonlinear integral inequalities. For this reason, the global existence of all solutions to (1.1) has to be ensured (we refer to [2] for a general discussion of the global existence issue) and this leads to conditions that are more restrictive than ours. Within our setting we allow certain solutions to blow-up in finite time, as one can see from the particular case 

\[f(t, u, u') = -\frac{(n + 1)(n + t)}{t^{n+2}} u^2 - \frac{2}{t^{2n}} u^3\]

with $n \geq 2$. In this case Theorem (2.3) applies despite the fact that the solution $u(t) = t^{\frac{n}{2-n}}$, $t \in [1, 2)$, blows-up in finite time.

### 3. Application to quasilinear elliptic equations

In this section, we shall apply the comparison method and Theorem (2.3) to prove that there exists a bounded positive solution to the quasilinear elliptic equation in two-dimensional exterior domains, 

\[(3.1) \quad \Delta u + f_1(x, u) + f_2(x, u) x \cdot \nabla u + f_3(x, u) (x \cdot \nabla u)^2 = 0, \quad |x| \geq 1.\]
We first recall the comparison method. Consider the elliptic equation
\[
\Delta u + \phi(x, u, \nabla u) = 0, \quad x \in G_A,
\]
where \( G_A = \{ x \in \mathbb{R}^2 : x \geq A \} \) for some \( A > 0 \). Fix some \( \alpha \in (0, 1) \). Let \( \phi \in C^0(M \times J \times \mathbb{N}, \mathbb{R}) \) for every bounded domain \( M \subset G_A \), every bounded interval \( J \subset \mathbb{R} \), and every bounded domain \( N \subset \mathbb{R}^2 \). Assume that for every bounded domain \( M \subset G_A \) there exists a nonnegative continuous function \( \theta_M \) such that
\[
|\phi(x, t, p)| \leq \theta_M(|t|)(1 + |p|^2), \quad x \in M, \; t \in \mathbb{R}, \; p \in \mathbb{R}^2.
\]
A solution \( u(x) \) of (3.2) in \( G_B \) for some \( B \geq A \) is defined to be a function \( u \in C^{2+\alpha}(M) \) for every bounded domain \( M \subset G_B \), such that \( u(x) \) satisfies (3.2) at every point \( x \in G_B \). A subsolution of (3.2) is defined to be a function \( u \) of the same regularity that satisfies \( \Delta u + \phi(x, u, \nabla u) \geq 0 \). Similarly, a supersolution of (3.2) satisfies \( \Delta u + \phi(x, u, \nabla u) \leq 0 \). Set \( S_B = \{ x \in \mathbb{R}^2 : x \geq B \} \) for \( B \geq A \).

The following lemma encompasses the version of the comparison method that will be used in the sequel.

**Lemma (3.3).** [11] Assume that \( \phi \) satisfies the above assumptions. If for some \( B \geq A \geq 0 \) there exists a positive subsolution \( w \) and a positive supersolution \( v \) to (3.2) in \( G_B \) such that \( w(x) \leq v(x) \) for all \( x \in G_B \cup S_B \), then (3.2) has a solution \( u \) in \( G_B \) such that \( w(x) \leq u(x) \leq v(x) \) throughout \( G_B \cup S_B \) and \( u(x) = v(x) \) for \( x \in S_B \).

We present now the main theorem of this section.

**Theorem (3.4).** Assume that there exists a number \( \alpha \in (0, 1) \) such that \( f_1, f_2, \) and \( f_3 \in C^0(M \times J, \mathbb{R}) \) for every bounded domain \( M \subset \mathbb{R}^2 \), every bounded interval \( J \subset \mathbb{R} \), and these functions satisfy the following conditions
\[
0 \leq f_i(x, t), \quad x \in G_1 \subset \mathbb{R}^2, \; t \in [0, \infty);
\]
\[
|f_i(x, u)| \leq F_i(|x|, |u|), \quad i = 1, 2, 3, \; x \in G_1 \subset \mathbb{R}^2, \; u \in \mathbb{R},
\]
where for every \( i = 1, 2, 3, \; F_i : [1, \infty) \times [0, \infty) \rightarrow [0, \infty) \) is Hölder continuous and \( F_i(r, s) \) is non-decreasing in \( s \) for every fixed \( r \in [1, \infty) \).

(i) If for some \( c > 0 \) we have
\[
\int_1^\infty s \max \{1, \ln s\} \left( F_1(s, 2c) + cs F_2(s, 2c) + (cs)^2 F_3(s, 2c) \right) ds < c
\]
then (3.1) has a bounded solution \( u \) in \( G_B \), with \( u(x) > 0 \) for \( |x| \geq 1 \).

(ii) If for some \( c > 0 \) we have
\[
\int_1^\infty s \ln s \left( F_1(s, 2c) + cs F_2(s, 2c) + (cs)^2 F_3(s, 2c) \right) ds < \infty
\]
then there is some \( B \geq 1 \) such that (3.1) has a bounded solution \( u \) in \( G_B \), with \( u(x) > 0 \) in \( G_B \).

**Proof.** We first prove part (i) of Theorem (3.4). Let us consider the following differential equation
\[
\Delta u + F_1(|x|, u) + F_2(|x|, u) |(x \cdot \nabla u)| + F_3(|x|, u)(x \cdot \nabla u)^2 = 0, \quad |x| \geq 1.
\]
The change of variables \( t = x \), \( u(x) = y(x) \), transforms the above equation into

\[
(3.8) \quad y''(t) + \frac{y'}{t} + F_1(t, y(t)) + tF_2(t, y(t)) \quad |y'| + i^2F_3(t, y(t))(y')^2 = 0, \quad t \geq 1.
\]

Applying Theorem (2.3), in view of hypothesis (3.6), we obtain that there exists a \( \delta \in (0, c) \) such that (3.8) has at least a bounded positive solution \( y(t) \) which satisfies \( \delta \leq y(t) \leq 2c - \delta \) for \( t \geq 1 \), and \( \lim_{t \to \infty} y(t) \) exists. If we set \( v(x) = y(t) \), then we have that \( \delta \leq v(x) \leq 2c - \delta \) for \( |x| \geq 1 \), \( \lim_{|x| \to \infty} v(x) \) exists, and

\[
\Delta v + f_1(x, v) + f_2(x, v)x \cdot \nabla v + f_3(x, v)(x \cdot \nabla v)^2
\leq \Delta v + F_1(\ |x|, v) + F_2(\ |x|, v) \quad x \cdot \nabla v | + F_3(\ |x|, v)(x \cdot \nabla v)^2
= y''(t) + \frac{y'}{t} + F_1(t, y(t)) + tF_2(t, y(t)) \quad |y'| + i^2F_3(t, y(t))(y')^2 = 0
\]

Hence, \( v(x) \) is a supersolution to (3.1) on \( |x| \geq 1 \). In addition, \( u(x) \equiv \delta \) satisfies obviously

\[
\Delta u(x) + f_1(x, u(x)) + f_2(x, u(x))x \cdot \nabla u(x) + f_3(x, u(x))(x \cdot \nabla u(x))^2 \geq 0
\]

for \( |x| \geq 1 \). The above inequality shows that \( u(x) \equiv \delta \) is a subsolution to (3.1) on \( |x| \geq 1 \). Applying Lemma (3.3) with \( B = A = 1 \), we deduce that there exists a solution \( u \) to (3.1) such that \( 0 < \delta \equiv u(x) \leq u(x) \leq v(x) \) for all \( |x| \geq 1 \), and \( u(x) = v(x) > 0 \) on \( S_1 \). This proves (i) of the theorem.

In addition, by part (ii) of Theorem (2.3) and the previous considerations, we deduce that the statement for (ii) of Theorem (3.4) is true. This completes the proof.

Example (3.9). For the equation in \( G_1 \cup S_1 \subset \mathbb{R}^2 \)

\[
\Delta u + \frac{u^2}{12(1 + |x|^2)^2} - \frac{x \cdot \nabla u}{3(1 + |x|^2)^3} - \frac{(x \cdot \nabla u)^2}{3(1 + |x|^2)^3} = 0,
\]

with \( F_1(\ |x|, \ u) = \frac{u^2}{12(1 + |x|^2)^2} \), \( F_2(\ |x|, \ u) = \frac{1}{3(1 + |x|^2)} \), \( F_3(\ |x|, \ u) = \frac{1}{3(1 + |x|^2)} \). A straightforward computation yields

\[
\int_1^\infty s \max \{1, \ln s\} \left( F_1(s, 2) + s F_2(s, 2) + s^2 F_3(s, 2) \right) ds < 1.
\]

Therefore, Theorem (3.4) ensures that the above equation has a bounded positive solution \( u(x) \) with \( u(x) > 0 \) for \( |x| \geq 1 \). Observe that the results from [3], [5], [11], [13], [16] are powerless.

Let us consider the particular case of (3.1),

\[
(3.10) \quad \Delta u + f(x, u) + g(x) x \cdot \nabla u = 0, \quad G_1 \cup S_1 \subset \mathbb{R}^2.
\]

As a consequence of Theorem (3.4), we have
Corollary (3.11). Assume that $g$ is class of $C^1$ and there exists a number $\alpha \in (0, 1)$ such that $f \in C^\alpha(M \times J, \mathbb{R})$ for every bounded domain $M \subset \mathbb{R}^2$, every bounded interval $J \subset \mathbb{R}$ and satisfies the following conditions

\[ 0 \leq f(x, k), \quad |x| \geq 1, \; k \in [0, \infty); \]
\[ |f(x, u)| \leq F(|x|, |u|), \quad |x| \geq 1, \; u \in \mathbb{R}, \]

where $F : [1, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is Hölder continuous and $F(r, s)$ is non-decreasing in $s$ for every fixed $r \in [1, \infty)$.

(i) If for some $c > 0$ we have
\[ \int_1^\infty s \max\{1, \ln s\} \left( F(s, 2c) + cs \right) ds < c \]
then (3.10) has a bounded positive solution $u(x)$ with $u(x) > 0$ for $|x| \geq 1$.

(ii) If for some $c > 0$ we have
\[ \int_1^\infty s \ln s \left( F(s, 2c) + cs \right) ds < \infty \]
then there is some $B \geq 1$ such that (3.10) has a bounded positive solution $u(x)$ with $u(x) > 0$ in $G_B \cup S_B$.

Example (3.12). For the equation in $G_1 \cup S_1 \subset \mathbb{R}^2$

\[ \Delta u + \frac{\sqrt{1 + u^2}}{6(1 + |x|^2)} - \frac{x \cdot \nabla u}{2(1 + |x|^2)^2} = 0, \]

with $F(|x|, |u|) = \frac{\sqrt{1 + u^2}}{6(1 + |x|^2)^2}$ and $g(x) = -\frac{1}{2(1 + |x|^2)}$, a straightforward computation yields
\[ \int_1^\infty s \max\{1, \ln s\} \left( F(s, 2) + s \right) ds < 1. \]

Therefore, Corollary (3.11) ensures that there exists a bounded positive solution $u(x)$ with $u(x) > 0$ for $|x| \geq 1$. Observe that the results from [3], [11], [13] are not conclusive.

Consider the particular case of (3.1),

(3.13) \[ \Delta u + f(x, u) = 0, \quad |x| \geq 1. \]

As a consequence of Theorem (3.4), we have

Corollary (3.14). Assume that $f$ is locally Hölder continuous in $(G_1 \cup S_1) \times \mathbb{R}$ and satisfies the following conditions

\[ 0 \leq f(x, k), \quad |x| \geq 1, \; k \in [0, \infty); \]
\[ |f(x, u)| \leq a(|x| |u|), \quad |x| \geq 1, \; u \in \mathbb{R}, \]

where $w(r)$ is non-decreasing for all $r \geq 0$, $a \in C((1, \infty), [0, \infty))$, and $w \in C((0, \infty), [0, \infty))$.

(i) If for some $c > 0$ we have
\[ \int_1^\infty s \max\{1, \ln s\} a(s)w(2c) ds < c \]
then (3.13) has a bounded positive solution $u(x)$ with $u(x) > 0$ for $|x| \geq 1$.
(ii) If for some $c > 0$ we have

$$\int_1^\infty s \ln s \alpha(s) \, ds < \infty$$

then there is some $B \geq 1$ such that (3.13) has a bounded positive solution $u(x)$ with $u(x) > 0$ in $G_B \cup S_B$.

**Example (3.15).** Among the equations of the form (3.13), we have the Emden-Fowler equation

$$\Delta u + p(x) \vert u \vert^r \text{sign}(u) = 0, \quad r > 0, \quad G_1 \cup S_1 \subset \mathbb{R}^2,$$

where $p(x)$ is nonnegative and Hölder continuous in $\mathbb{R}^n$.

(i) For the sublinear ($0 < r < 1$) or superlinear ($r > 1$) Emden-Fowler equations, if

$$\int_1^\infty s \max\{1, \ln s\} \max_{|x|=s} \{p(x)\} \, ds < \infty,$$

then there exists $c > 0$ large enough or $c > 0$ small enough, such that

$$\int_1^\infty s \max\{1, \ln s\} \max_{|x|=s} \{p(x)\} \, |x|^r \, ds < c.$$  

Consequently, applying Corollary (3.14), we deduce that if (3.16) holds, then the sublinear and the superlinear Emden-Fowler equation has a bounded solution $u$ in $G_1$, with $u(x) > 0$ in $G_1 \cup S_1 \subset \mathbb{R}^2$.

(ii) For the linear Emden-Fowler equation ($r = 1$), if

$$\int_1^\infty s \max\{1, \ln s\} \max_{|x|=s} \{p(x)\} \, |x| \, ds < \frac{1}{2},$$

then, applying Corollary (3.14), we obtain the existence of a bounded solution $u$ in $G_1$, with $u(x) > 0$ in $G_1 \cup S_1 \subset \mathbb{R}^2$.

Note that under the same conditions (3.16) or (3.17), the investigations in [3], [11], [13] show only that there is some $B > 1$ such that the Emden-Fowler equation has a solution $u$ in $G_B$, with $u(x) > 0$ for $|x| \geq B$. □

**Remark:** Our approach is typically 2-dimensional ($n = 2$). For the $n$-dimensional case with $n \geq 3$, we refer to [14], [15]. □

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