ON THE CHARACTERIZATION OF A SUBVARIETY OF SEMI-DE MORGAN ALGEBRAS

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Abstract. In this note we characterize by a new set of axioms the largest subvariety of semi-De Morgan algebras with the congruence extension property.

1. Introduction

The equational class of semi-De Morgan algebras was introduced by Sankappanavar in [8]. It consists of bounded distributive lattices with an additional unary operation and it contains the variety of pseudocomplemented distributive lattices and $K_{1,1}$, one of the subvarieties of Ockham algebras which includes De Morgan algebras.

In [4] Hobby developed a duality for semi-De Morgan algebras which he used to find the largest subvariety of semi-De Morgan algebras with the congruence extension property. This variety, which Hobby denoted by $C$, contains both $K_{1,1}$ and the equational class of demi-pseudocomplemented lattices, a generalization of pseudocomplemented lattices studied by Sankappanavar in [9] and [10].

The equations defining principal congruences as well as the subdirectly irreducibles of the variety $C$ were determined by us in [6]; however, the two inequalities ($\alpha$ and $\beta$) that characterize this subvariety of semi-De Morgan are rather complicated. In fact Problem 2 in [4] is to find "nicer axioms for $C$".

We solved this problem algebraically determining a new inequality ($\gamma$) such that $C$ can be characterized by $\gamma$ and $\beta$.

2. Preliminaries

We start by recalling some definitions and essential results from [8].

Definition (2.1). An algebra $L = (L, \vee, \wedge, ', 0, 1)$ is a semi-De Morgan algebra if the following five conditions hold ($a, b \in L$):

(S1) $(L, \vee, \wedge, 0, 1)$ is a distributive lattice with 0, 1.
(S2) $0' \approx 1$ and $1' \approx 0$.
(S3) $(a \vee b)' \approx a' \wedge b'$.
(S4) $(a \wedge b)'' \approx a'' \wedge b''$.
(S5) $a''' \approx a'$.

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This equational class of algebras will be denoted by $SDMA$.

The following rules hold in $SDMA$ and some of them are proved in [8]:

(S6) $(a \land b)' \approx (a'' \land b'')' \approx (a \land b')'$.

(S7) $(a \land b)' \approx (a' \lor b')''$.

(S8) $(a \land b)''' \approx (a' \lor b')'$.

(S9) $a \leq b$ implies $b' \leq a'$.

(S10) $a \land (a \land b)' \geq a \land b'$.

(S11) $(a \lor b)''' \approx (a' \land b')' \approx (a'' \lor b'')' \approx (a \lor b'')''$.

3. The variety $\mathcal{C}$

D. Hobby determined in [4] the largest subvariety of $SDMA$ with the congruence extension property. He characterized this variety, which he denoted by $\mathcal{C}$, by the following inequalities:

$$(a) \quad a' \lor b' \geq (a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)'$$

$$(b) \quad a' \lor (a' \land b \land b')' \geq (a \land b)'.$$  

It is possible to obtain simpler inequalities characterizing $\mathcal{C}$. The search for these inequalities requires some rather nasty calculations so we consider several lemmas before we can reach our goal.

With this aim we will consider first the following identities:

$$(\alpha_1) \quad a' \lor b' = a' \lor b' \lor ((a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)'')$$

$$(\beta_1) \quad a' \lor (a' \land b \land b')' = (a \land b)' \lor (a' \land b \land b')'.$$

These identities are equivalent to $\alpha$ and $\beta$, respectively (note that $a \geq a \land b$ implies $a' \leq (a \land b)'$).

Now we can prove the following.

**Lemma (3.1).** Let $L \in \mathcal{C}$ and $a, d \in L$. Then the identity $\alpha_1$ implies

$$(a_2) \quad (a' \lor d') \land d'' = (a \land d')' \land d''.$$

**Proof.** By (S3), $(a' \land d'') \lor d' = (a \lor d')' \lor d'$. Replacing $b$ by $a \lor d'$, $a$ by $d$ and $c$ by $d'$ in the identity $\alpha_1$ and using commutativity, we obtain

$$(a \lor d')' \lor d'$$

$$(a \lor d')' \lor d' \lor ((a \lor d')' \land d' \land d'' \land ((a \lor d')' \land d' \land d'' \land ((a \lor d')' \land d' \land d''))$$

$$(a \lor d')' \lor d' \lor ((a \lor d')' \land d) \land d' \land d''$$

because $((a \lor d')' \land d'')' = ((a \lor d')' \land d)$ by (S6).
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But \((d \land d')' \geq ((a \lor d') \land d')'\) since \(d \land d' \leq (a \lor d') \land d'\); hence it follows from the previous equation that

\[
(a \lor d')' \lor d' = (a \lor d')' \lor d' \lor (((a \lor d') \land d')') \\
= (a \lor d')' \lor d' \lor (((a \lor d') \land d')')' \quad \text{(by S3)} \\
= (a \lor d')' \lor d' \lor ((a \lor d') \land (d \lor d'))' \quad \text{(by distributivity)} \\
= d' \lor (a \lor d') \land (d \lor d'))' \quad \text{(because \((a \lor d')' \leq ((a \lor d') \land (d \lor d'))')} \\
= d' \lor ((a \lor d) \land d')' \quad \text{(by distributivity)} \\
= d' \lor ((a \lor d) \land d').
\]

Therefore we have \((a' \lor d'') \lor d' = ((a \lor d') \land d'') \lor d'\).

Now, by distributivity, we obtain

\[
(a' \lor d') \land (d'' \lor d') = ((a \lor d') \land d') \land (d'' \lor d').
\]

Since \((a \land d')' \geq d'\), it follows that

\[
(a' \lor d') \land (d'' \lor d') = (a \lor d)' \land (d'' \lor d')
\]

and meeting the two members with \(d''\), we have \(a_2\). □

**Lemma (3.2).** Let \(L \in \text{SDMA}\) and let \(a, b, c \in L\). Then

\[
\begin{align*}
(\alpha_2) & \quad (a' \lor b') \land b'' = (a \lor b')' \land b'' \quad \text{and} \\
(\beta_1) & \quad a' \lor (a' \land b \lor b')' = (a \land b)' \lor (a' \land b \lor b')'
\end{align*}
\]

imply

\[
(\delta) \quad (a' \land (b \land (c \lor c'))')' \lor (b' \land (a \land c')) = (a \land b)' \land (a \land c)' \land ((b \land (c \lor c'))').
\]

**Proof.** Let us denote by \(A\) and \(B\), respectively, the left and right sides of the identity \(\delta\). We are going to prove that \(\beta_1\) and \(\alpha_2\) imply \(A = B\) using the distributivity of \(L\).

First we will verify that the joins of \(A\) and \(B\) with \((a' \land c' \land c'')'\) are equal:

\[
A \lor (a' \land c' \land c'')' = (a' \land (b \land (c \lor c'))')' \lor ((b' \lor (a' \land c' \land c'')') \land ((a \land c)' \lor (a' \land c' \land c'')'))
\]

(by distributivity)

\[
= (a' \land (b \land (c \lor c'))')' \lor ((b' \lor (a' \land c' \land c'')') \land (a' \lor (a' \land c' \land c'')))
\]

(by \(\beta_1\) and S6)

\[
= (a' \land (b \land (c \lor c'))')' \lor (b' \land a') \lor (a' \land c' \land c'')' \quad \text{(by distributivity)}
\]

\[
= (a' \land (b \land (c \lor c'))')' \lor (a' \land c' \land c'')'
\]

because, by S9, \((b \land (c \lor c'))' \geq b'\) and thus \(a' \land (b \land (c \lor c'))' \geq a' \land b'\).

\[
B \lor (a' \land c' \land c'')' = ((a \land b)' \land (b \land (c \lor c'))')' \lor (a' \land c' \land c'')' \land ((a \land c)' \lor (a' \land c' \land c''))
\]

(by distributivity)
Therefore, denoting by $d$ the expression $a \lor c \lor c'$, we will have

$$(a' \land c' \land c'')' = d'' \geq d'$$

and thus

$$A \land (a' \land c' \land c'')' = A \land d''$$

$$= ( (a' \land (b \land (c \lor c'))') \lor (b' \land (a \land c')) ) \land d''$$

$$= ( (a \lor (b \land (c \lor c'))) \lor (b' \land (a \land c')) ) \land d''$$ (by S3)

$$= ( (a \lor b) \land (a \lor c \lor c') \land d'') \lor ( (b' \land (a \land c')) \land d'' )$$ (by distributivity)

$$= ( (a \lor b) \land d' \land d'' ) \lor ( (b' \land (a \land c')) \land d'' )$$ (by the definition of $d$ , )

$$= ( (a \lor b') \land d' \land d'' ) \lor ( (b' \land (a \land c')) \land d'' )$$ (by $a_2$)

$$= ( (a' \land b') \land d' \land d'' ) \lor ( (b' \land (a \land c')) \land d'' )$$ (by S3)

$$= ( (a' \land b') \land d' \land (b' \land (a \land c')) \land d'' )$$ (by distributivity)

$$= (d' \lor (b' \land (a \land c')) \land d''$$ (because $a \land c' \geq a'$)

$$= (b' \land (a \land c') \land d'') \lor (d' \land d'')$$ (by distributivity)

$$= (b' \land (a \land c') \land d'') \lor d'$$ (because $d'' \geq d'$).

By a similar process,

$$B \land (a' \land c' \land c'')' = B \land d''$$

$$= (a \land b' \land (b \land (c \lor c'))') \land d'' \land (a \land c')$$ (by commutativity)

$$= ((a \land b) \lor (b \land (c \lor c'))) \land d'' \land (a \land c')$$ (by S3)

$$= (b \land (a \lor c \lor c')) \land d'' \land (a \land c')$$ (by distributivity)

$$= (b \land d') \land d'' \land (a \land c')$$ (by the definition of $d$ )

$$= (b' \land d') \land d'' \land (a \land c')$$ (applying $a_2$)
and, by distributivity, this identity is equivalent to
\[ \alpha (3.3), \text{these are equivalent to} \]
which is equivalent to
\[ \alpha \]
hold.

Finally, by the distributivity of \(L\), we conclude that \(A = B\).

**Lemma (3.3).** Let \(L \in SDMA\) and let \(a, b, c \in L\). Then the identity
\[
(\delta) \quad (a' \land (b \land (c \lor c'))') \lor (b' \land (a \land c')) = (a \land b)' \land (a \land c)' \land ((b \land (c \lor c'))')'.
\]
is equivalent to \(\alpha\).

**Proof.** First note that \(\alpha\) is equivalent to
\[
(a' \lor b') \land (a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)' = (a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)' \land (b \land c)''
\]
and, by distributivity, this identity is equivalent to
\[
(a' \land (a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)'') \lor
(b' \land (a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)'') =
(a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)''.
\]

By S9, it is known that \(a'\) is less than or equal to \((a \land b)'\) and to \((a \land c)'\), and that \(b'\) is also less than or equal to \((a \land b)'\), \((b \land c)'\) and \((b \land c)'\). Therefore the previous identity is equivalent to
\[
(a' \land (b \land c)' \land (b \land c)'') \lor (b' \land (a \land c)) = (a \land b)' \land (a \land c)' \land (b \land c)' \land (b \land c)''
\]
and, by S3, also to
\[
(a' \land ((b \land c) \lor (b \land c)')) \lor (b' \land (a \land c)) = (a \land b)' \land (a \land c)' \land ((b \land c) \lor (b \land c)')'.
\]
Finally, by the distributivity of \(L\), we conclude that \(\alpha\) is equivalent to \(\delta\).

From the previous lemmas we obtain the following:

**Proposition (3.4).** Let \(L \in SDMA\). Then \(L \in C\) if and only if the identities
\[
(\alpha_2) \quad (a' \lor b') \land b'' = (a \land b)' \land b''
\]
and
\[
(\beta_1) \quad a' \lor (a' \land b \land b')' = (a \land b)' \lor (a' \land b \land b')'
\]
hold.

**Proof.** We proved in Lemma 3.1 that the identity \(\alpha_2\) is a consequence of \(\alpha_1\) which is equivalent to \(\alpha\).

Conversely, by Lemma (3.2), \((\alpha_2\) and \(\beta_1\)) imply \((\delta\) and \(\beta_1\)), and by Lemma (3.3), these are equivalent to \(\alpha\) and \(\beta_1\).
It is now possible to characterize $\mathcal{C}$ by simpler axioms solving Problem 2 in Hobby [4]:

**Theorem (3.5).** The subvariety $\mathcal{C}$ of semi-De Morgan algebras can be characterized by inequalities $\gamma$ and $\beta$:

1. \(\gamma\) \hspace{1cm} a' \lor b' \geq (a \land b)' \land b''
2. \(\beta\) \hspace{1cm} a' \lor (a' \land b \land b')' \geq (a \land b)'\]

**Proof.** It is enough to prove that the identity $\alpha_2$ of the previous lemma is equivalent to the inequality $\gamma$.

By $\alpha_2$ we have

\[a' \lor b' \geq (a' \lor b') \land b'' = (a \land b)' \land b''.\]

Therefore $\alpha_2$ implies $\gamma$.

On the other hand, from $\gamma$ we know that

\[(a' \lor b') \land (a \land b)' \land b'' = (a \land b)' \land b''.\]

But $a' \leq (a \land b)'$ and $b' \leq (a \land b)'$ so that $a' \lor b' \leq (a \land b)'$ and therefore $\alpha_2$ follows from $\gamma$. \qed

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