A NOTE ON ASYMPTOTIC INTEGRATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The paper is concerned with the asymptotic behavior of solutions to a second order nonlinear differential equation \( u'' + f(t, u) = 0 \). Using the Banach contraction principle, we establish global existence of solutions which satisfy \( u(t) = At + o(t^\nu) \) as \( t \to +\infty \), where \( A \in \mathbb{R} \) and \( \nu \in (0, 1) \).

1. Introduction

Asymptotic behavior of solutions of nonlinear second order differential equations
\begin{equation}
    u'' + f(t, u, u') = 0, \quad t \geq t_0 \geq 1 
\end{equation}
and
\begin{equation}
    u'' + f(t, u) = 0, \quad t \geq t_0 \geq 1 
\end{equation}
has always been the subject of intensive research. Many papers published recently are concerned with existence of solutions to Eqs. (1.1) and (1.2) which behave at infinity like solutions of the simplest second order differential equation, \( u'' = 0 \), see, for instance, [1]-[9], [11]-[22]. A thorough study of the properties of such solutions, called asymptotically linear [5] or linear-like [16], is important, for instance, for the theory of oscillation of ordinary and functional differential equations, see the references in [9], as well as for the study of existence of positive solutions of elliptic problems in exterior domains, cf. [2] and [21]. We also note that this type of asymptotic behavior has been addressed recently by the authors in connection with Weyl’s limit circle and limit point classification of differential operators in the theory of singular Sturm-Liouville problems [10].

Two particular types of behavior of asymptotically linear solutions of Eqs. (1.1) and (1.2) have been studied more extensively. Namely, Constantin [1], Rogovchenko and Rogovchenko [16], Yin [21] and Zhao [22] explored conditions which guarantee asymptotic representation
\begin{equation}
    u(t) = At + o(t) \quad \text{as } t \to +\infty, 
\end{equation}
whereas Lipovan [5], Mustafa [8] and the authors [9] established conditions for a more precise asymptotic development
\begin{equation}
    u(t) = At + B + o(1) \quad \text{as } t \to +\infty, 
\end{equation}
for some real constants \( A \) and \( B \).
Using a fixed point argument and a Wronskian-type representation similar to those exploited in [11], [12], the first author established recently in [8] existence of solutions of Eq. (1.2) which, for a given $\mu \in (0, 1)$, have asymptotic representation

\[(1.5) \quad u(t) = At + o(t^\mu) \quad \text{as } t \to +\infty.\]

As pointed out by Lipovan [5] and the authors [9], asymptotic formula (1.3) embraces large classes of solutions to Eq. (1.2), including those satisfying (1.4) or, in case this is not possible, solutions with the asymptotic representation (1.5).

It is known that Eq. (1.2) may possess solutions with the asymptotic development (1.4) in some situations where standard results on asymptotic integration guarantee only existence of solutions that behave at infinity as (1.3) or, at most, as (1.5), see the details in our paper [9, pp. 364-365]. Furthermore, a class of solutions with asymptotic representation (1.5) contains also solutions which satisfy (1.4). Therefore, in order to complete the study of solutions with asymptotic expansions (1.3)-(1.5) and understand completely relationship between all three classes, it is natural to explore existence of solutions of Eq. (1.2) that can be expressed in the form (1.5), but do not satisfy (1.4). The first attempt to answer this question has been made by the authors in [13]. To simplify the formulation of the result we adapt from the cited paper, we introduce two constants

\[\theta(n, t_0) := \int_{t_0}^{+\infty} s^n a(s) ds \quad \text{and} \quad \gamma := t_0^{\delta-(1+\epsilon)c} \theta(m + (1 + \epsilon)c, t_0).\]

Application of [13, Theorem 2.2] to the celebrated Emden-Fowler equation

\[(1.6) \quad u'' + a(t)|u|^m \text{sgn } u(t) = 0, \quad t \geq 1, \quad m \geq 1,\]

frequently encountered in applications, leads to the following proposition.

**Theorem (1.7).** Let $c \in (0, 1)$, $\epsilon \in (0, c^{-1} - 1)$, $\delta \in (c, (1 + \epsilon)c)$ and let $a(t)$ be a continuous, nonnegative function that does not vanish eventually. Assume also that

(i) $\theta(m + (1 + \epsilon)c, 1) < +\infty$;
(ii) $\theta(m, t_0) < cm^{-1}$;
(iii) $\theta(m + (1 + \epsilon)c, t_0) < ct_0^{\delta}$.

Then, for every $A \in (0, 1 - \gamma(ct_0^{\delta})^{-1})$, there exists a solution $u(t)$ of Eq. (1.6) with the asymptotic representation

\[(1.8) \quad u(t) = At + w(t) \quad \text{as } t \to +\infty,\]

where $w(t) = o(t^{1-\delta})$ and, for all $t \geq t_0$,

\[A^m \left[ \int_{t_0}^{t} s^{m+(1+\epsilon)c} a(s) ds + \int_{t}^{+\infty} s^{m+(1+\epsilon)c-1} a(s) ds \right] \leq w(t) \leq \frac{\gamma}{(1+\epsilon)c} t_0^{\delta(1+\epsilon)c-\delta} t^{1-(1+\epsilon)c}.\]

It is clear that for the solution $u(t)$ whose existence is established in Theorem (1.7), one has

\[\liminf_{t \to +\infty} w(t) \geq B = A^m \gamma t_0^{\delta(1+\epsilon)c-\delta} > 0,\]
which, however, does not rule out the possibility that $u(t)$ has the asymptotic development (1.4).

In this note, using a modification of the Hale-Onuchic technique [4] that has been successfully applied by the first author [7] to investigate asymptotic behavior of solutions with prescribed decay of the first derivative, we establish, under rather general assumptions, existence of global solutions to Eq. (1.2) that satisfy (1.5) and can be written in the form (1.8), where

$$\lim_{t \to +\infty} w(t) = +\infty,$$

which, obviously, excludes for these solutions possibility of asymptotic representation (1.4).

2. Asymptotic behavior of solutions

**Theorem (2.1).** Let $A > 0$, $\nu \in [0, 1)$, $u_0 \in \mathbb{R}$ and $\alpha, \beta \in C([t_0, +\infty); [0, +\infty))$ be two functions such that $\alpha(t) \leq \beta(t)$ for all $t \geq t_0$ and $\beta(t) = o(t^{-\nu})$ as $t \to +\infty$.

Introduce the set $D_{A,u_0}$ by

$$D_{A,u_0} = \{ u \in C^1([t_0, +\infty); \mathbb{R}) \mid \alpha(t) \leq u'(t) - A \leq \beta(t) \quad \text{for all } t \geq t_0, \quad u(t_0) = u_0 \},$$

and assume that for all $t \geq t_0$ and $u \in D_{A,u_0}$,

$$\alpha(t) \leq \int_t^{+\infty} f(s, u(s))ds \leq \beta(t).$$

Suppose further that for all $t \geq t_0$ and any $u_1, u_2 \in D_{A,u_0}$,

$$|f(t, u_1(t)) - f(t, u_2(t))| \leq \frac{k(t)}{t} |u_1(t) - u_2(t)|,$$

where a function $k \in C([t_0, +\infty); [0, +\infty))$ satisfies

$$\int_{t_0}^{+\infty} k(t)dt < 1 - \nu.$$

Then there exists a unique solution of the initial value problem

$$\begin{cases} u'' + f(t, u) = 0, & t \geq t_0 \geq 1, \\ u(t_0) = u_0, \end{cases}$$

defined on $[t_0, +\infty)$ such that

$$u(t) = At + o(t^{1-\nu}) \quad \text{as } t \to +\infty,$$

$$\alpha(t) \leq u'(t) - A \leq \beta(t), \quad t \geq t_0.$$

If, in particular,

$$\int_{t_0}^{+\infty} \alpha(t)dt = +\infty,$$

one has

$$\lim_{t \to +\infty} |u(t) - At| = +\infty.$$
Proof. Define the distance between the functions $u_1$ and $u_2$ in $D_{A,u_0}$ by
\[ d(u_1, u_2) = \sup_{t \geq t_0} \left[ t^\nu |u_1(t) - u_2(t)| \right]. \]
Then the metric space $E = (D_{A,u_0}, d)$ is complete. For $u \in D_{A,u_0}$ and $t \geq t_0$, introduce the operator $T : D_{A,u_0} \to C^1([t_0, +\infty); \mathbb{R})$ by the formula
\[ (Tu)(t) = u_0 + At - t_0 + \int_{t_0}^t \int_{s}^{+\infty} f(\tau, u(\tau))d\tau ds. \]
It is not hard to see that $T$ is well-defined, that is, $TD_{A,u_0} \subseteq D_{A,u_0}$. Furthermore, we shall prove that $T$ is a contraction in $D_{A,u_0}$. Let
\[ \lambda = \frac{1}{1 - \nu} \int_{t_0}^{+\infty} k(s)ds. \]
It follows from the estimate
\[
|\textbf{(Tu)}_1'(t) - \textbf{(Tu)}_2'(t)| \leq \int_{t}^{+\infty} \frac{k(\tau)}{s} |u_1'(\tau) - u_2'(\tau)| d\tau \\
\leq \int_{t}^{+\infty} \frac{k(s)}{s} \int_{t_0}^{s} |u_1'(\tau) - u_2'(\tau)| d\tau ds \\
\leq \left( \int_{t}^{+\infty} \frac{k(s)}{s} \int_{t_0}^{s} \frac{1}{s} d\tau ds \right) d(u_1, u_2) \\
\leq t^{-\nu} \left( \frac{1}{1 - \nu} \int_{t_0}^{+\infty} k(s)ds \right) d(u_1, u_2) = t^{-\nu} \lambda d(u_1, u_2)
\]
that, for $u_1, u_2 \in D_{A,u_0}$,
\[ d(Tu_1, Tu_2) \leq \lambda d(u_1, u_2). \]
By virtue of (2.2), $\lambda \in (0, 1)$, and the existence of a solution follows now from the Banach contraction principle. Furthermore, for all $t \geq t_0$,
\[ u(t) - At = u_0 - At_0 + \int_{t_0}^{t} \int_{s}^{+\infty} f(\tau, u(\tau))d\tau ds \geq u_0 - At_0 + \int_{t_0}^{t} a(s)ds, \]
which yields
\[ \lim_{t \to +\infty} \left[ u(t) - At \right] = \int_{t_0}^{+\infty} a(t)dt = +\infty. \]
The proof is complete. \qed

Application of Theorem (2.1) to Emden-Fowler equation (1.6) leads to the proposition which complements results established in [3, 7, 13, 20]. In what follows, $C := t_0^{-(p+\varepsilon)}\theta(m + \nu + \varepsilon, t_0)$, where $\theta$ is defined as above.

**Corollary (2.4).** Let $\nu \in [0, 1)$, $\varepsilon \in (0, 1 - \nu)$, and let $a(t)$ be a continuous, nonnegative function that does not vanish eventually. Assume that
\begin{enumerate}
  \item[(a)] $\theta(m + \nu + \varepsilon, 1) < +\infty$;
  \item[(b)] $\theta(m, t_0) < m^{-1} (1 - \nu)$;
  \item[(c)] $\theta(m + \nu + \varepsilon, t_0) < t_0^{+\varepsilon}$.
\end{enumerate}
Then, for every \( A, \) \( 0 < A < 1 - C, \) there exists a solution \( u(t) \) of Eq. (1.6) with the asymptotic representation (1.8), where \( w(t) = o(t^{1-\nu}) \) as \( t \to +\infty \) and, for all \( t \geq t_0, \)

\[
A^m \int_{t_0}^{t} \int_{s}^{+\infty} \tau^m a(\tau) d\tau ds \leq w(t) \leq \int_{t_0}^{t} \int_{s}^{+\infty} \tau^m a(\tau) d\tau ds.
\]

In particular, \( w(t) \) satisfies (1.9) provided that

\[(d) \; \theta(m + 1, 1) = +\infty.\]

**Proof.** Let \( u_0 = At_0. \) For \( t \geq t_0, \) introduce the functions \( \alpha(t) \) and \( \beta(t) \) by

\[
\alpha(t) = A^m \int_{t}^{+\infty} s^m a(s) ds \quad \text{and} \quad \beta(t) = \int_{t}^{+\infty} s^m a(s) ds.
\]

Taking into account (2.3) and the fact that

\[
\left( \frac{t}{t_0} \right)^{\nu+\varepsilon} \beta(t) \leq t_0^{-(\nu+\varepsilon)} \int_{t_0}^{+\infty} s^{m+\nu+\varepsilon} a(s) ds \leq C,
\]

we deduce that for all \( t \geq t_0 \) and all \( u \in D_{A,A_0}, \)

\[
\alpha(t) \leq \int_{t}^{+\infty} a(s) \left( As + \int_{t_0}^{s} a(\tau) d\tau \right) m ds \leq \int_{t}^{+\infty} a(s) |u(s)|^m ds
\]

\[
= (Tu)'(t) - A \leq \int_{t}^{+\infty} a(s) \left( As + \int_{t_0}^{s} \beta(\tau) d\tau \right) m ds
\]

\[
\leq \int_{t}^{+\infty} a(s) \left( As + \int_{t_0}^{s} C \left( \frac{t_0}{\tau} \right)^{\nu+\varepsilon} d\tau \right) m ds
\]

\[
\leq \int_{t}^{+\infty} s^m a(s)(A + C)^m ds \leq \beta(t).
\]

Furthermore, for any \( u_1, u_2 \in D_{A,A_0} \) and for all \( t \geq t_0, \) one has

\[
|f(t, u_1(t)) - f(t, u_2(t))| = t^m a(t) \left| \left( \frac{u_1(t)}{t} \right)^{m} - \left( \frac{u_2(t)}{t} \right)^{m} \right|
\]

\[
\leq \frac{mt^m a(t)}{t} \sup_{s \geq t_0} \left[ \left( \frac{1}{s} \left( As + \int_{t_0}^{s} \beta(\tau) d\tau \right) \right)^{m-1} \right] |u_1(t) - u_2(t)|
\]

\[
\leq \frac{k(t)}{t} |u_1(t) - u_2(t)|,
\]

where \( k(t) = mt^m a(t). \) The conclusion follows now from Theorem (2.1). \( \Box \)

We conclude the paper by noticing that it is not difficult to see that, given two positive constants \( c_1 < c_2, \) any continuous function \( a(t) \) such that

\[
c_1 t^{-m-2} \leq a(t) \leq c_2 t^{-m-2}, \quad t \geq 1,
\]

will satisfy conditions (a)-(d) of Corollary (2.4).
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